Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00020-X
J. Appl. Maths Mechs, Vol. 65, No. 1, pp. 171–173, 2001

0021–8928/01/$—see front matter

THE STEADY ROLLING OF A DISC ON A ROUGH PLANE†

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(*Received* 21 *March* 2000)

The well-known problem of the rolling without slipping of a heavy circular disc along a horizontal plane is considered. The steady motions of a disc for which the angle between the plane of the disc and the supporting plane (the angle of nutation) is constant are investigated. The problem of the range of variation of the angle of nutation within which the given motions are stable, irrespective of the values of the constants of the two linear first integrals or, in other variables, irrespective of the angular velocities of precession and proper rotation, is investigated. It is shown that this range is wider than was established earlier in [1]. © 2001 Elsevier Science Ltd. All rights reserved.

The steady motions of a disc for which the angle of nutation is constant are termed [1] steady rollings (SRs). Earlier [1–5], the conditions of their existence and stability were obtained by different methods. It was noted in [4] that SRs form a two-dimensional manifold. The purpose of the present paper is to distinguish on this manifold the regions in which the known conditions of stability are certainly satisfied.

The equations of motion of a disc on a perfectly rough plane admit of three first integrals: the energy integral

$$2H = (A_1 + ma^2)\dot{\theta}^2 + A_1 q^2 + (A_3 + ma^2) r^2 + 2mga\sin\theta = 2h = \text{const}$$

and two linear integrals, specified implicitly by the relations (everywhere henceforth summation over the indices i and j is carried out from 1 to 2)

$$r = \sum_i u_i c_i, \quad u_i = F\left(\xi, \eta, 1; \frac{1 + (-1)^i \cos\theta}{2}\right), \quad c_i = \text{const}, \quad i = 1, 2 \tag{1}$$

$$q = \frac{A_3}{2A_1}\sin\theta \sum_i (-1)^{i+1} \upsilon_i c_i, \quad \upsilon_i = F\left(\xi + 1, \eta + 1, 2; \frac{1 + (-1)^i \cos\theta}{2}\right)$$

and having the form of hypergeometric series.

Here, m is the mass of the disc, a is its radius, A_1 and A_3 are respectively the equatorial and axial moments of inertia, g is the acceleration due to gravity, θ is the angle of nutation, q and r are the projections of the angular velocity of the disc onto the line of maximum slope and onto the normal to the plane of the disc, and F is the hypergeometric function, the parameters ξ and η of which are roots of the quadratic equation

$$s^2 - s + B = 0, \quad B = \frac{ma^2 A_3}{A_1(A_3 + ma^2)}$$

Below, we will assume that these parameters are complex conjugate quantities, i.e. that the following inequality holds

$$B > 1/4 \tag{2}$$

Let W be the minimum of the function H with respect to the variables $\dot{\theta}$, r and q on the level of the c_1 and c_2 integrals, specified implicitly by relations (1). Hence

$$W = \frac{1}{2}\sum_{i,j} \left[(A_3 + ma^2) u_i u_j + (-1)^{i+j} \frac{A_3^2}{4A_1} \sin^2\theta\, \upsilon_i \upsilon_j\right] c_i c_j + mga\sin\theta$$

The expression for W can be rewritten in dimensionless form

†*Prikl. Mat. Mekh.* Vol. 65, No. 1, pp. 173–175, 2001.

$$W = \frac{1}{2}\sum_{i,j} w_{ij}x_i x_j + \sin\theta, \quad w_{ij} = (2k+1)u_i u_j + (-1)^{i+j} k \sin^2\theta \, v_i v_j$$

$$A_1 = kma^2, \quad A_3 = 2kma^2, \quad x_i = c_i\sqrt{a/g}, \quad i = 1,2$$

Inequality (2) then becomes the inequality $k < 3.5$.

It is well known (see [1–5]) that the disc is able to execute SR, i.e. motions specified by the relations

$$\theta = \alpha = \text{const}, \quad \dot\theta = 0, \quad q = \text{const}, \quad r = \text{const}$$

The angle α is determined from the equation $dW|d\theta|_{\theta=\alpha} = 0$. It was shown in [1] that any SR is stable if the angle θ lies in the range $(0; \pi/4]$. Detailed analysis of the conditions of stability enable us to simplify this result.

The equation $dW|d\theta|_{\theta=\alpha} = 0$ can be written in explicit form as follows:

$$\sum_{i,j} a_{ij}x_i x_j - \cos\alpha = 0 \tag{3}$$

$$a_{ij} = a_{ji} = \sin\alpha[(k+\tfrac{1}{2})((-1)^i u_j v_i + (-1)^j u_i v_j) + (-1)^{i+j} k\cos\alpha \, v_i v_j]$$

For each fixed $\alpha \neq \pi/2$, Eq. (3) gives a hyperbola, but when $\alpha = \pi/2$ it gives a pair of intersecting straight lines [6].

The condition for SR to be stable obtained using the modified Routh Salvadori theorem [5], has the form $d^2W/d\theta^2|_{\theta=\alpha} \geqslant 0$, or

$$\sum_{i,j} b_{ij}x_i x_j - \sin\theta \geqslant 0 \tag{4}$$

$$b_{ij} = b_{ji} = 2(2k+1)u_i u_j + (3k+\tfrac{1}{2})\cos\alpha((-1)^i u_j v_i + (-1)^j u_i v_j) +$$
$$+ \ (-1)^{i+j}((k+1)\sin^2\alpha + 3k\cos^2\alpha)v_i v_j$$

For each fixed α, if $k > 1/\sqrt{3} - 1/2$, the boundary of the region of stability is an ellipse, the stable region lying outside this ellipse and the unstable region within it.

Thus, if, for some fixed α, the hyperbola (3) and the ellipse (4) do not intersect, then steady rolling, for which $\theta = \alpha$, will be stable irrespective of the values of $x_i (i = 1, 2)$.

We will investigate this problem in more detail. From Eq. (3) we express the variable x_2 (for $\alpha < \pi/2$ this is possible for any x_1) and substitute it into inequality (4). We obtain

$$\sum_i [i(-1)^i a_{3-i,2}(a_{1i}b_{22} - a_{22}b_{1i})]x_i^2 + a_{22}G \geqslant$$

$$\geqslant \pm 2(a_{12}b_{22} - a_{22}b_{12})x_1\sqrt{(a_{12}^2 - a_{11}a_{22})x_1^2 + a_{22}\cos\alpha} \tag{5}$$

$$G = G(\alpha) = (b_{22}\cos\alpha - a_{22}\sin\alpha)$$

By direct checking it can be proved that the coefficient of x_1^2 on the left-hand side of the inequality in the range $\alpha \in (0; \pi)$ is positive. The quantity a_{22} is also positive in this range. The expression for $G(\alpha)$ in the range $(0; \alpha_1)$ acquires a positive values, while in the range $(\alpha_1; \pi)$ it acquires negative values. Here, $\alpha_1 = \alpha_1(k) < \pi/2$ is the root of the equation $G(\alpha) = 0$.

Thus, in the range $(0; \alpha_1)$, on the left-hand side of inequality (5) there is obviously a positive expression, which enables both sides of this inequality to be squared. The following biquadratic inequality is obtained.

$$p_1 x_1^4 + p_2 x_1^2 + p_3 \geqslant 0 \tag{6}$$

$$p_1 = 2(2k+1)^2(k+1)(2k+\sin^2\alpha)\sin^2\alpha(u_2 v_1 + u_1 v_2)^4 > 0$$

$$p_2 = \left[(2k+1)^2 G\cos 2\alpha - \frac{Sa_{22}}{\sin\alpha}\right]\frac{\sin\alpha}{\cos\alpha}(u_2 v_1 + u_1 v_2)^2, \quad p_3 = G^2 \geqslant 0$$

$$S = S(\alpha) = [(2k+3)^2 + 3(2k+1)^2]\cos^4\alpha - 4(2k+1)(4k+3)\cos^2\alpha + (2k+1)^2$$

Inequality (6) holds for any value of x_1 if one of the following two conditions is satisfied:

(a) the discriminant of the corresponding biquadratic equation is negative;

(b) $p_2 > 0$.

By direct checking it can be shown that condition (a) is equivalent to the condition $S < 0$, or in explicit form

$$\cos^2\alpha > \cos^2\alpha_2 = \frac{2(2k+1)[(4k+3)-\sqrt{6(2k+1)(k+1)}]}{(2k+3)^2+3(2k+1)^2} \tag{7}$$

Furthermore, both analytically and by numerical experiment it can be shown that, for any k from the interval considered, the inequality

$$\alpha_3 < \alpha_2 < \alpha_1$$

holds. Here, $\alpha_3 = \alpha_3(k)$ is the root of the equation $p_2(\alpha) = 0$, where $p_2(\alpha) > 0$ when $\alpha < \alpha_3$.

Consequently, the SR is stable when $\alpha \in (0; \alpha_2]$, i.e. for all α satisfying inequality (7). For example, for a homogeneous disc ($k = 1/4$) we have

$$\cos^2\alpha > (24-9\sqrt{5})/38 \approx 0{,}102$$

It can be seen that the given range of variation of α is wider than the interval $(0; \pi/4)$.

I wish to thank A. V. Karapetyan for his interest.

This research was supported financially by the Russian Foundation for Basic Research (98-01-00041, 00-15-96150).

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Translated by P.S.C.